Evolution of an approximate analysis technique for unbraced steel frames

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Synopsis
The fundamental principles of a new approximate method of analysis for unbraced rigid frames are presented. The method makes use of a multi-curve interaction principle placing the failure load on the one hand and a load related to the elastic buckling load on the other hand. The failure curves between these extreme values are empirical.

In this study they have been tested against 19 steel frame structures from previous research and 11 model frames specifically designed for the purpose of this investigation. In general, a better correlation with exact solutions was recorded than could be obtained from existing similar interaction techniques. Considering the simple application of the procedure, notably if the required elastic analysis components are further approximated, the method could become a useful tool for the engineer when designing sway frames.

Introduction
Plastic design of unbraced frames is complicated by the need to make adequate allowance for the loss in load carrying capacity induced by in-plane instability effects of which the $P-\Delta$ moments are the most prominent.

In general, past research has attempted to solve the problem of instability in three ways, namely:
1. By using rigorous analysis techniques taking the exact moment-curvature characteristic into account.
2. By using an elastic-plastic iteration procedure on the frame as a whole or on suitable subassemblies.
3. By using interaction formulae applied to the entire framework.

This paper describes the development of an alternative approximate technique for the elasto-plastic analysis of unbraced rigid frames. The proposed technique, which allows for the treatment of single portal frames as well as for multi-storey and multi-bay structures, is not confined to steel to which it has been applied in this paper but could also be developed for other materials such as reinforced concrete.

In essence, the method represents a refinement and extension of the Merchant-Rankine interaction formula. A modified version of the Merchant-Rankine formula has been proposed by the European Convention for Constructional Steelwork (ECCS) and was also intended for inclusion in the new British Steel Design Code, BS. The existing interaction formula predicting the approximate failure load of unbraced frames are simple in their application when compared with other methods of analysis. However, they have a number of restrictions and shortcomings which have been illustrated in the relevant literature. The new method attempts to eliminate these shortcomings without adding unduly to the complexity of the evaluation.

The proposed analysis technique

Single column analogy
Classical column buckling analysis is based on column curves such as shown in Fig 1.

The concept embodied in Fig 1 can be transferred to entire frameworks. In this case $P_{yield}$ will have to be replaced by the plastic collapse load, $P_R$, and the elastic buckling load $P_E$ of the column becomes the elastic buckling load, $P_0$, of the entire framework. The 'limiting slenderness ratio', $\lambda_0$, of the column becomes the 'limiting slenderness ratio' of the so-called 'limiting frame'. The geometry of the 'limiting frame' is such that first yield at a critical section and elastic failure coincides. Frames other then the 'limiting frame' will be identified by the ratio of their elastic buckling load to plastic collapse load $P_0/P_R$. The derivation and nature of the limiting frame concept is discussed in detail by way of an example in the next section.

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General principles of new technique
The new method takes the format of a multi-curve interaction approach as shown in Fig 2. Each curve represents the same behaviour of an infinite number of frames, all of which have the same 'limiting slenderness ratio'. In this paper, initially, the computation and meaning of the various parameters associated with the new technique are briefly summarized. A detailed description of how the individual curve shape was derived is subsequently demonstrated by way of an example. In addition, the results of frames analysed by this method are briefly discussed.

To analyse a frame in accordance with the interaction curves of Fig 2, the so-called reduction factor $\alpha$ and the two ratios $aP_0/P_R$ and $(aP_0/P_R)l$ need to be known. The latter ratio of elastic buckling load to plastic collapse load refers to the 'limiting frame'. Each curve of Fig 2 represents a certain group of frames characterized by the same parameter $(aP_0/P_R)l$. Thus for a given frame the ratio $(aP_0/P_R)l$ identifies the relevant failure curve from the array of possible curves of Fig 2, whereas the slope $aP_0/P_R$ enables the location of the actual structure on the selected curve. The factor $\alpha$ is equal to 1.0 for completely symmetrical structures and loadings and reduces to less than 1.0 in accordance with Eqn 1 for any deviation from full symmetry. The full justification for Eqn 1 is set out in Appendix 1.

$$\alpha = \frac{0.4}{1 - 0.6 \left(\frac{P_0}{P_R}\right)^{1.2}}$$

Fig 1: Column curve

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The term $\lambda_{HO.4}$ is representative of the non-symmetrical conditions and takes the format of a slenderness ratio. It is calculated from Eqn 2
using moments and axial forces applicable to the non-symmetrical load components related in magnitude to 0.4 $P_O$. In addition $P_f$ is set equal to 250N/mm² and $\gamma_r$ equal to 1.0. The value $\lambda_{HO.4}$ corresponding to the smallest ratio $\lambda/s_{HO.4}$ found for the structure is significant, where $\lambda$ is the slenderness ratio of member.

The derivation of Eqn 2 appears in Appendix II.

$$\lambda_{2} = \frac{\lambda_{1} M_{L}}{r P_f} \left[ \frac{1 + \frac{1}{2} \left( \sqrt{1 + \frac{4}{9} P_f} \right)}{1 + \frac{1}{2} \left( \sqrt{1 + \frac{4}{9} P_f} \right)} \right]$$

where

- $\lambda_{2}$ = 'limiting slenderness ratio' of 'limiting frame'
- $\lambda$ = distance from centroid of section to extreme fibre
- $r$ = radius of gyration
- $M$ = elastic second-order bending moment
- $P_f$ = axial member force
- $L$ = member length
- $I$ = moment of area
- $P_f$ = stress at onset of yield

For the non-symmetrical case, Eqn 2 is evaluated twice, first for $\alpha=0.4$ to obtain $\lambda_{HO.4}$ to calculate the final value for $\alpha$ from Eqn 1, then for the final value of $\alpha$, which will give $\lambda_{2}$. For structures with relatively low axial forces the term in brackets is close to 1.0 so that it will suffice to evaluate the expression in front of the brackets. This applies to many practical building frames.

The term $\lambda_{2}$ is defined as the 'limiting slenderness ratio' of a 'limiting frame'. This frame, if subjected to a load arrangement equal to the factored applied loading but in magnitude related to its own elastic buckling load would just reach first yield ($P_f$) at the critical section. The 'limiting frame' signifies the transition from completely elastic failure to inelastic failure.

Actual frames of a specific frame group and their 'limiting frame' are related by Eqn 3.

$$\left( \frac{a_{P_f}}{a_{P_f}} \right) = R \frac{\alpha_{P_{O}}}{a_{P_f}} \frac{\lambda}{\lambda_{2}}$$

The validity of Eqn 3 is demonstrated on the example presented in the next section.

The coefficient $R$ recognizes that the reduction of the fully-plastic moment capacity due to axial forces may be different for the actual frame and its 'limiting frame'. However, usually $R=1.0$ will give a satisfactory result. For the structure as a whole the lowest ratio $\lambda/\lambda_{2}$ is significant. To determine the failure load of a frame in accordance with the above approach would thus require a first-order rigid plastic analysis to obtain $P_f$, an elastic buckling analysis for $P_O$ and a second-order load analysis to find $\alpha$ and $\lambda$. There are many approximate procedures to deal with the elastic analyses so that modified first-order approaches will often suffice.

Evolution of design curve shape

Procedure

Initially, the load-slenderness curves for a number of frames will be drawn for regular increments of the slenderness ratio $\lambda$. Subsequently, these curves will be converted into plots fitting a non-dimensional graph such as presented in Fig 2.

The principle is first demonstrated on the uniform symmetrical portal frame of Fig 3. For this frame the graph development procedure will comprise four consecutive stages. The frame of Fig 3 will be identified within each of the four stages and the status of the various parameters will be defined at the beginning of each stage. Thereafter, the method will be extended to include for horizontal loading and other changes.

The following two hypotheses need to be confirmed:

1. A group of frames as defined by Eqn 3 can be presented on a common curve in a non-dimensional interaction diagram as shown in Fig 2.
2. On a particular failure curve, a frame of lower slenderness will have a higher failure load as related to its actual plastic collapse load than a more slender frame.

Vertical loading - 'Symmetry-buckling'

Fig 2: Basic interaction curves

Stage 1:
Constant parameters for the purpose of the example:
- $I$, $A$, $E$, $l_r$, $f_y$
Constant parameters for any one curve:
- $(m_\alpha EA)$ for a particular $EA$ and base fixity, $n$, $P_O$

Variable parameters for any one curve:
- $\lambda$, $P_O$, $h$, $L=nh$

These parameters apply to Fig 4 and to a problem such as given in Fig 3.

Following the Euler principle, the elastic buckling load of a frame can be expressed as $P_O = m_\alpha EA/\lambda^2$ where $m_\alpha$, which is a function of $L/h$, for the given problem, reflects the elastic buckling load of the complete frame. This hyperbolic function is presented in Fig 4 for frames of different geometry. Each curve corresponds to a particular ratio $n=L/h$, which is given as a ratio of length to the point on the relevant curve marked thus $\lambda$. For a particular value of slenderness ratio $\lambda$, the curves only differ by the factor $m_\alpha$, which in turn remains unchanged for any one curve when $\lambda$ varies. The variation in $\lambda$ is thereby modified by the member lengths $h$ and $L$. Hence each curve of Fig 4 represents a whole group of frames. For any one group of frames with a specific geometry factor $n=L/h$, the buckling load, axial member forces; bending moments and thus stresses can be expressed as a function of the term $m_\alpha EA/\lambda^2$. Recognizing that for given base and loading conditions, $m_\alpha$ is solely a function of $n$, the stress at a particular section can be...
expressed as follows:

\[ \text{stress} = f(n, \frac{EA}{k^2}) \]

If the frame geometry and its parameters are known and if, furthermore, the stress is equated to the stress value at the onset of yield, the above expression can be solved for a specific value \( \lambda \), called the 'limiting slenderness ratio' \( \lambda_0 \). This parameter is well known in individual column design and has hence been applied to frames in a similar way.

The slenderness ratios \( \lambda_0 \) and the corresponding points on the buckling curve are marked thus \( \bullet \) in Fig 4. They indicate the transition from elastic to inelastic failure, and the associated failure load would equal \( P_{0L} = \frac{m_0}{h^2} \frac{m_0}{h^2} \). The corresponding frame has been defined as the 'limiting frame'.

Example frame of Fig 3: The buckling parameter \( m_0 = 1.48 \) has been obtained from a computer analysis for a frame with a ratio of beam to column length \( n = 3 \). From this follows the buckling load of the actual frame

\[ P_{0L} = P_{0L} = \left( \frac{m_0}{h^2} \right)^2 \frac{E}{h} = 868 \text{ kN} \]

The critical section of the frame is section 1-1 of Fig 3. Loading the frame of Fig 3 with \( P_{0L} = 868 \) kN results in the following bending moment for the critical section using a second-order elastic analysis:

\[ M_{1,1} = 69.5 \text{ kNm} \]

Using Eqn 2 the 'limiting slenderness ratio' \( \lambda_0 = 752 \). The 'limiting frame' will have the properties and configuration of the actual frame, with the only difference that

\[ h_0 = \frac{h}{\lambda} = 8.69 \text{ m} \]

\[ L_0 = \frac{L}{\lambda} = 26.08 \text{ m} \]

This result is now put to the test. It needs to be shown that the 'limiting frame' attains a stress of 250 MPa at section 1-1, when subjected to its own elastic buckling load \( P_{0L} \). The buckling load for the 'limiting frame' amounts to

\[ P_{0L} = P_{0L} = \left( \frac{m_0}{h^2} \right)^2 \frac{E}{h} = 1.28 \text{ kN} \]

The value \( m_0 = 1.48 \) is also applicable to the 'limiting frame'. The bending moment at section 1-1 of the 'limiting frame' due to \( P_{0L} = 1.28 \) kN has been calculated as

\[ M_{1,1} = 2.66 \text{ kNm} \]

For this moment, the corresponding stress at section 1-1 becomes

\[ \sigma_{1,1} = \frac{M_{1,1}}{2A} \frac{E}{h} = 250 \text{ MPa} \]

It can easily be verified that other frames, similar to the portal frame of Fig 3, would yield the same 'limiting slenderness ratio' of \( \lambda_0 = 752 \), eg retaining \( n = 3 \) but changing the length \( L \) to 0.75m, or keeping \( L \) at 1.0m but changing the cross-section to 50x50mm, will also result in \( \lambda_0 = 752 \). All these frames will have a different \( P_{0L} / P_0 \) ratio and \( \lambda_0 \) changes.

At the same time it can be shown that in Eqn 3 the ratio \( (P_{0L} / P_0) \) of the corresponding 'limiting frames' remains unaltered. This proves that an infinite number of frames with a common 'limiting frame' may be accommodated on a particular interaction curve, such as that given in Fig 2. Furthermore, a specific frame of a certain group of frames is distinguishable by its ratio \( P_{0L} / P_0 \), which in turn is proportional to the inverse of the column slenderness \( \lambda \).

The easiest way to generate a specific group of frames is to modify the absolute member lengths \( L \) and \( h \) but keeping their ratio \( n \) constant so that the frames are geometrically similar. For example the frame of Fig 3 the results for \( P_{0L} \) and \( \lambda_0 \) are marked on Fig 4. The buckling load \( P_{0L} \) of the actual frame would fall outside the graph for the scale chosen in Fig 4. The relevant curve is therefore shown again in the inset to Fig 4.

Stage 2:

Constant parameters for the purpose of the example:

\[ I, A, r, E, I_y, M_p \]

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**Fig 4:** Elastic buckling load of frames

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THE CIVIL ENGINEER in South Africa — December 1984

589
Constant parameters for any one curve:
\[ L_0 = L, \quad \lambda_0, \quad \eta_1, \quad P_F = P_{P_0}, \quad P_{O_0}, \quad m_{O_0}, \quad n_0. \]

Variable parameters for any one curve:
\[ h, \quad \lambda, \quad m_1, \quad P_O, \quad P_F, \quad n = L/h. \]

These parameters apply to Figs 5 and 6 and to a problem such as given in Fig 3.

From stage 1, only the transition points, again shown thus \( \bigcirc \), have been transferred to another diagram depicting the frame failure load \( P_F \) on the vertical axis (Fig 5). These transition points correspond to the 'limiting frame' which is characterized by a dimension ratio \( n_0 = L_0/\lambda_0 \), given in Fig 5 and the 'limiting slenderness ratio', \( \lambda_0 \). A particular curve of Fig 5 is obtained by keeping the beam length constant at a size corresponding to that of the 'limiting frame', ie \( L = L_0 = n_0 \lambda_0 r. \)

At the same time, the column height is allowed to vary from zero to infinity. As a result, \( L \) is different for different curves since \( n_0 \) and \( \lambda_0 \) changes and, in addition, for any one curve the ratio \( L/h \) varies from zero to infinity because of the height change.

With reference to the constant parameters stipulated for this stage, individual curves will have a constant ratio \( L/r \) and plastic collapse load \( P_{P_0} = 16 \text{Mp}/L. \) Due to the variation in \( L \), different curves will show proportionally different \( P_F \) values on the vertical axis. At the limiting conditions the plastic collapse load would be \( P_{P_0} = P_P \), and the elastic buckling load would equal \( P_{O_0} = m_2 \eta_1 E / \lambda_0^2 \), as already derived for stage 1.

The curve section between the 'limiting slenderness ratio' \( \lambda_0 \) and \( \lambda = 0 \) represents frames failing by inelastic instability. In this region the curve shapes of Fig 5 are empirical.

For this research it has been assumed that the curves branch-off tangentially from the exact elastic buckling curves, applicable to frames with slenderness ratios greater than the 'limiting slenderness ratio' \( \lambda_0 \). At the other end, near \( \lambda = 0 \), the failure curves of frames would gradually reach the plastic collapse load in a near-tangential approach. Similar curve shapes were suggested by \( \text{Lu}^{15} \) based on an exact elasto-plastic analysis for pinned-base portal frames. \( \text{Scholz}^{16} \) confirmed the general curve shapes by testing four small-scale portal frames with constant beam length but varying column height.

Curves of the nature assumed in Fig 5 will eventually lead to the final interaction graph proposed in this research. These interaction curves were compared with a number of discrete theoretical solutions and results of various laboratory tests. A good agreement was obtained.

It needs to be mentioned that frames falling on a particular curve of Fig 5 correspond to different 'limiting frames' by definition of Eqn 3. In fact, in simple terms, Fig 5 represents a series of curves for frames with different beam and column stiffnesses in which each curve represents all the frames with the same beam length and fully plastic collapse load, but not the same \( \lambda_0 \) or \( P_{O_0} \).

Points on the curves of Fig 5 other than the transition points can be identified by referring to the relevant 'limiting slenderness ratio'. This is demonstrated by obtaining some points on the curve passing through the transition point marked \( \bigcirc \), for which \( L/r = 1024. \)

**Elastic range**: To find a point \( (x, y) \) on the curve of Fig 5, corresponding to a slenderness ratio \( \lambda = 1024 \), it is necessary that the column height is double that of the 'limiting frame', for which \( \lambda_0 = 512 \) and \( n_0 = 2 \). The modified frame column will thus attain a slenderness ratio of 1024 and \( n \) would change to 1. Returning to Fig 4, the corresponding point for \( \lambda = 1024 \) is found on the curve for \( n = 1 \) and has been marked thus \( \bullet \). The relevant failure load, which is elastic, will equal \( m_0 E / A \times 1024 \), where \( m_0 \) refers to a column with \( N = 1 \).

**Inelastic range**: Points in the inelastic range such as \( \text{a} \) can be related to column heights by simple proportion from the limiting condition, ie the column height and slenderness ratio corresponding to point \( \bullet (n = 3) \) would have to be \( n_0 / n = 2 / 3 \) times those at the limiting condition, ie \( \lambda = 341 \) applies. The failure load \( P_F \) applicable to the frame with \( \lambda = 341 \) is evaluated from empirical curves, as discussed previously in this section.

**Example frame of Fig 3**: Frames with the same ratio \( n = L/h \) of Fig 5 are found on different curves. By definition of Eqn 3 such frames have the same 'limiting frame' and 'limiting slenderness ratio.' For the frame of Fig 3, \( n = 3 \) and \( L/h = 3 \), \( \lambda = 2256 \), the corresponding 'limiting frame' with \( \lambda_0 = 752 \) is marked thus \( \bigcirc \) in Fig 5.

A summary of results applicable to this 'limiting frame' is given below:

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![Figure 5: Failure loads of frames](image-url)

**Fig 5**: Failure loads of frames

\[ P_F = 64 \text{kn} \]
\[ \lambda = 26.83 \]
\[ \lambda = 216.3 \]
\[ \lambda = 1024 \]

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THE CIVIL ENGINEER in South Africa — December 1984
\[ M = \frac{1}{4} d^3 = 4 \text{KNm} \]

\[ P = \frac{16\text{M}_p}{L_1} = \frac{15 \times 4}{26.08} = 2.45 \text{ kN} \]

\[ m_o = 1.48 \]

\[ P_o = \frac{m_o \rho}{\lambda_{11}^2} = \frac{1.48 \rho}{\lambda_{11}^2} \]

\[ \frac{P_o}{P_p} = \frac{1.48}{\lambda_{11}^2} \]

The plastic collapse load for the actual frame of Fig 3 is given by

\[ P = \frac{16\text{M}_p}{L} = 64 \text{ kN} \]

To identify the failure load of this actual frame on curves such as given in Fig 5, it is argued that a curve will exist for which \( L/r = 86.5 \). In this instance, the beam length and the plastic collapse load will both coincide with that of the actual frame of Fig 3.

The condition described above may come about for a curve such as shown in the inset to Fig 5. On this particular curve the actual frame with \( \lambda = 28.83 \) can be located by proportional reference to the relevant transitional slenderness ratio of 216, which corresponds to a value \( n = 0.4 \). The failure load \( P_F \) of the actual frame is indicated on the same curve.

Stage 3: For stage 3, the same definition of parameters applies as for stage 2.

This stage involves the simple transfer of all curves and the points marked on the curves of Fig 5 to a new plot giving the non-dimensional ratio \( P_F/P_p \) on the vertical axis rather than the absolute failure load \( P_F \) (Fig 6). As a result, all curves of Fig 6 run into the common intersection point \( P_F/P_p = 1.0 \) on the vertical axis. The constraint of constant beam length for the individual curve of Fig 5 throughout the \( \lambda \)-range still applies.

For the example frame of Fig 3 the position of the actual frame with \( L/r = 86.5 \) as well as the ratio \( (P_0/P_p)_{\lambda} \) are indicated on the relevant curves of Fig 6. It must be reiterated that by definition of Eqn 3 the actual and 'limiting' frames are not located on the same curve. This will only be accomplished in the final interaction graphs.

Stage 4:

Constant parameters for the purpose of the example:

\( l, A, r, I, \gamma, M_p \).

Constant parameters for any one curve:

\( \lambda, h, L_1, L_3, h_1, h_2, L/h, n, P_0, P_p, (P_0/P_p)_{\lambda} \)

Variable parameters for any one curve:

\( \lambda, P_0, h, P_p, L, L/h \). These parameters apply to Fig 7 and a problem such as given in Fig 3.

In stage 4 the construction of the final interaction plot of Fig 7 will be explained for frames with a common dimensional ratio \( n = 3 \). For this purpose, it is necessary to identify the points on the different \( L/r \)-curves of Fig 6 which correspond to the same value of \( n = L/h = 3 \), i.e. points corresponding to \( \lambda = h/r = (l/in) (L/r) \). Hence point 3 on the curve for \( L/r = 1024 \) corresponds to \( \lambda = 1/3 \times 1024 \).

Likewise, other points 3 have been indicated on all curves of Fig 6. The relevant point on the curve for \( L/r = 3185 \) is found on the elastic curve section. For the curve applicable to \( L/r = 2256 \), the transitional point is directly valid. Points on the curves for \( L/r \) less than 2256 will fall in the inelastic region.

It is possible to change the horizontal axis to represent the dimensionless ratio \( P_F/P_p \) in Fig 7, by noting that the slope of any radial line through the origin equals \( P_F/P_p \). Thus any point on Fig 6, corresponding to a specific value of \( n \), can be transferred onto Fig 7 by recording the relevant value of \( P_F/P_p \) (vertical axis of Fig 6) and calculating the magnitude of \( P_0/P_p \) for the associated value of \( \lambda \) (horizontal axis of Fig 6). The scale of the slope of the radial line through the origin is determined from the slope of the line through the limiting values \( (P_0/P_p) \). The ratio of these slopes is inversely proportional to the relevant slenderness ratios for frames with the same geometric parameters. In this way points 3 on the curve of Fig 7 can be developed from equivalent points 3 of Fig 6.

The resulting plot is taken as the common locus of all frames with a dimensional ratio \( n = 3 \) for portal frames with a uniformly distributed beam load, pinned bases, equal column and beam properties \( I, A, r \) and shape factor. For such frames, the buckling parameter \( m_o \) remains.

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*Fig 6: Failure load to plastic collapse load*
The actual elastic buckling load $P_O$, below the actual elastic buckling load $P_P$, will be selected, with the reduction factor $\alpha$ closely related to the 'limiting slenderness ratio'. Hence, for combined loading elastic failure curves would deviate by a factor $\alpha$ from the hyperbolic curves of Fig. 4. For the general load case, failure curves such as shown in Figs. 5 and 6 cannot be plotted since individual curves were based on a constant beam length $L$ throughout the $L$-range. As a result, the plastic collapse load $P_P$ remained unaltered as the column height was changed. If a horizontal load were added, the plastic collapse load $P_P$ would increase as the column height diminishes. However, the constant of constant beam length was finally removed for the interaction graph of Fig. 7. In fact, the intermediate plots of Figs. 5 and 6 were only presented here to facilitate the understanding of the development process behind the final interaction curves.

The supposition that a curve, such as that in Fig. 7, is also applicable to combined loading is based on the following two conditions:

Firstly, that frames with the same dimensional properties $n=L/h$, particular geometric properties $I, A, E$ and subjected to a specific load arrangement could be classified into a common group of frames at the limiting conditions. This would satisfy Eqn. 3 for specific values $m_Q, E, f_y, d/r$ and shape factor. Secondly, that frames on a particular interaction curve would display a progressively increasing failure load $P_P$ as their slenderness reduces.

Both these points have been confirmed by elastic second-order computer solutions, which were processed in a similar way as described for the case of 'symmetry-buckling'. The second statement has also been checked by appropriate laboratory testing and the relevant results are fully detailed in Reference 11.

The extended $n$-ratio to cover the variations in sectional properties and strength: If the ratio $n=L/h$ is extended to include sectional variations between the members, e.g. $n$ becomes $(L/h) (I_C/I_g)$, or allowance is being made for a difference in beam to column buckling or beam to column strength, this will not be reflected in the elastic buckling loads $P_O$ and $P_O$. It must be remembered that the inertia ratio is in itself not significant since the ratios $L/h$ and $I_C/I_g$ are interchangeable with respect to the elastic buckling loads as well as the corresponding parameters $m_Q$ and $m_O$.

However, the above variations will influence the plastic collapse loads $P_P$ and $P_P$, the reduction factor $\alpha$ and the 'limiting slenderness ratio' $\lambda_L$. The behaviour summarized in this conclusion confirms the two hypotheses formulated previously. It is also easy to verify that in Fig. 7 different curves would be obtained for different dimensional ratios $n$. The principal difference between these curves is the ratio $(P_O/P_P)$ of their 'limiting frames'.

Non-symmetrical conditions: In many ways a frame subjected to arbitrary loading behaves similarly to a frame subjected to vertical, symmetrical loading. As implied in an earlier section, for cases involving general loading (e.g. horizontal loading) a fictitious 'elastic buckling load'

\[
\frac{P_O}{P_P} = \frac{868.64}{26.08} = 26.08, \quad \lambda = \frac{752}{26.08} = 28.83
\]

Both slopes, i.e. $P_O/P_P$ and $(P_O/P_P)_L$ and the failure load ratio $P_F/P_P$ have been entered in Fig. 7.

\[
\begin{align*}
\frac{P_F}{P_P} & = 0.52 \\
\frac{P_O}{P_P} & = 0.52 \\
\frac{P_F}{P_P} & = 0.52
\end{align*}
\]

\[
\begin{align*}
\frac{P_F}{P_P} & = 0.52 \\
\frac{P_O}{P_P} & = 0.52 \\
\frac{P_F}{P_P} & = 0.52
\end{align*}
\]
The change in the plastic collapse load \( P_P \) is absorbed by the non-dimensional axis parameter \( P_{P}/P_P \) in graphs such as given in Fig 7. The other variations can only produce a deflection from the basic curve shape which is shown as a bold line in Fig 7.

A similar effect occurs, when different structural or load arrangements are compared. In fact, these influences can also change the elastic buckling parameters \( m_{O} \) and \( m_{OQ} \). Notwithstanding these modifications, the two principal hypotheses previously formulated are confirmed.

The phenomena described above, together with the wide variety of possible load applications, base conditions and subassemblage shapes makes it impractical to present curves such as that in Fig 7 as a function of the dimensional ratio \( n \). A vast number of different curve shapes would be necessary to cover all these eventualities for a given value of \( n \).

Hence, more conveniently, the ratio \((\alpha P_{O}/P_P)\) at the limiting conditions is proposed as the dominant variable. This ratio appears on the right, vertical axis of the non-dimensional diagram of Figs 2 or 7. Thus, for a particular frame \((\alpha P_{O}/P_P)\) has to be evaluated to select the curve applicable to the problem. Conveniently, the ratio \((\alpha P_{O}/P_P)\) can be determined in the way as already shown in Eqn 3 or from first principle.

The value \( \alpha \), which is related to the 'limiting slenderness ratio', will be derived from Eqn 1 once Eqn 2 has been evaluated for \( \alpha = 0.4 \). As stated earlier, for pure vertical load and symmetrical conditions \( \alpha = 1.0 \), so that Eqn 1 need not be evaluated for such cases.

**Comparisons with tests and rigorous analysis**

To confirm the empirical interaction curves in Fig 2 a wide selection of frames were examined and compared with rigorous elasto-plastic analysis as well as results of laboratory tests. More than 80 frames were investigated for this purpose ranging up to 16 stories in height and three bays in width. Cases of pure vertical load as well as combined vertical and horizontal load were considered. In all instances the Merchant-Rankine load was also computed. Included in this comparative study were 11 small-scale model frames which were designed with the objective to monitor two specific interaction curves of Fig 2.

In all instances good agreement with exact solutions could be reported when using the new interaction procedure. The difference in these cases not exceeding three per cent. On the other hand, the Merchant-Rankine rule has been found to vary by up to 89 per cent in isolated cases. Full details regarding these comparisons are presented in Reference 11.

**Summary and conclusion**

In this paper an alternative, approximate second-order, elasto-plastic method of analysis has been developed for unbraced building frames.

The method is based on an interaction approach and incorporates the plastic collapse load and certain elastic parameters. The principal elastic parameters are the elastic buckling load and the 'limiting slenderness ratio'. The 'limiting slenderness ratio' defines a 'limiting frame' similar to the actual frame. When subjected to loading related to its elastic buckling load, this 'limiting frame' is assumed to fail by reaching first yield at a critical section. The failure load of the 'limiting frame' constitutes one boundary in the interaction procedure, the second boundary being the plastic collapse load of the actual frame. Both values are taken as the two extreme points of a specific failure curve applicable to a particular group of frames. The failure load of the actual frame can be identified on the failure curve by a location parameter which incorporates properties of the actual frame.

The hypothesized inelastic portions of these failure curves have been tested against a number of discrete theoretical solutions and were compared with laboratory results from previous research as well as model frames especially designed for the purpose of this investigation. In all cases a satisfactory correlation between tests, theory and the proposed design approach was obtained.

The principal objective of this research was, to formulate a simple analysis method which would improve on the results calculated by the Merchant-Rankine rule, which is known to be conservative in many cases. It is believed that the proposed method has achieved this objective without adding unduly to the complexity of the procedure as a whole. Comparing both methods with exact results has shown that the new approach predicts the actual failure load of a frame more accurately than the Merchant-Rankine formula.

An added advantage of the method developed in this research is the possibility to adopt approximate analytical approaches to obtain the necessary elastic parameters and to present these in graphical form. In this way many common frame problems could be analysed without a computer, once the plastic collapse load is known.

In summary, it has been concluded that the method developed in this research appears to be a satisfactory substitute for a strict theoretical full-scale, elasto-plastic computer analysis. In this study the new method has been applied to steel structures, although the same principles could also be used for other materials.

Further research is presently directed towards the incorporation of working load deflections, in-plane member instability, lateral torsional buckling, further \( P-\Delta \) effects, residual stresses and the application of the method to three-dimensional systems and to reinforced concrete structures in general.

**Appendix I — factor \( \alpha \)**

**Symmetry-buckling**

In this case which refers to perfect conditions in regard to symmetry of loading and geometry the 'limiting slenderness ratio' \( \lambda_{l} \) is based on the full buckling load \( P_{O} \). Hence \( \alpha = 1 \). The value \( \lambda_{l} \) is associated with the bifurcation point which exists in the elastic load — rotation curve shown in Fig 8.

**Non-Symmetrical conditions**

For this case, which would apply to non-symmetrical structures, non-symmetrical loading or imperfect symmetrical structures, elastic analysis would reveal a progressive lateral collapse and no bifurcation of equilibrium would exist. The collapse curves, when shown in a load-displacement diagram, would approach the elastic buckling load of the structure in an asymptotic way. As for 'symmetry-buckling', the curve shapes would vary with the stiffness of the structure and the disposition and composition of the applied loading. Three arbitrary curves are shown in Fig 9 (solid lines).

All curves such as shown in solid lines in Fig 9 would reach infinite displacements \( D \) at the level of the elastic buckling load \( P_{O} \). Thus, if the 'limiting slenderness ratio' were directly related to the elastic buckling load \( P_{O} \), in this case \( \lambda_{l} \) would theoretically approach infinity for all non-symmetrical cases. As shown in Appendix II, the 'limiting slenderness ratio' is directly related to the second-order forces which in turn are a function of the displacement vector \( D \).

By implication, the value \( \lambda_{l} \) corresponds to a slope \((P_{O}/P_{O}) = 0 \) at all times by virtue of Eqn 3. In order to overcome this problem in this research, a somewhat lower load than \( P_{O} \) is designated as the fictitious 'elastic buckling load' of the structure. For this load, which is called \( \alpha P_{O} \) \( (0 < \alpha < 1) \), a finite displacement state and thus 'limiting slenderness ratio' can be identified as shown in Fig 9. It is significant to realize that, generally, for a structure of limiting conditions the ratio of \((\alpha P_{O}/P_{O})\) is now not equal to zero, since \( \lambda_{l} \neq \infty \).

To determine a suitable reduction value \( \alpha \), the following points were considered:

1. The formulation of \( \alpha \) would have to contain a parameter taking stiffness and loading into account.
2. The load level \( \alpha P_{O} \) should lead to rational values for the 'limiting slenderness ratio'.

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**Fig 8: 'Symmetry-buckling'**
Fig 9: Non-symmetry failure

3. A generalized approach covering all possible cases would be desirable.

After a thorough investigation it was felt that curve "C" of Fig 9 fulfills the requirements stipulated above. This curve, passing through \( aP_0 \), has been carefully chosen to link points on the load-displacement curves just before the deformations start to increase rapidly, thereby rendering the structure undesirable. Numerous variations to curve "C" of Fig 9 were also tested by computer but found to be less efficient than the curve obtained from the expression given by Eqn 1. The formula for \( a \) has yielded satisfactory results in terms of the 'limiting slenderness ratio' for a large number of frames analyzed by computer.

Appendix II — Limiting slenderness ratio

For a typical symmetrical cross-section, the maximum fibre stress allowing for bending, \( M \), and axial compression, \( P \), can be expressed as follows:

\[
\sigma_{\text{max}} = \sigma = \frac{P + M y}{A}
\]

The dimension \( y \) is the distance from the centre of gravity of the section to the extreme fibre.

Substituting the relationships \( A = l/r^2 \), \( r = L/r \) and \( P = m^2 EI/L^2 \) one obtains

\[
l_{\text{max}} = E \left( \frac{m^2 + m^2 y^2}{2 \bar{P} L^2} \right)
\]

Dividing by \( E \) and multiplying by \( r^2 \) leads to a quadratic equation for \( \lambda_k \), if subsequently \( 1/\lambda_k \) is substituted for \( l \):

\[
\frac{1 + \lambda_k M L}{1 + \lambda_k M L} \left( \frac{P^2}{l_{\text{max}}} \right) = \frac{P^2}{l_{\text{max}}}
\]

This expression has the solution given below:

\[
\lambda_k = \frac{M L}{r f_{\text{max}}} \left[ \frac{1 + \lambda_k M L}{2} \sqrt{\frac{1 + \lambda_k M L}{\frac{4 l_{\text{max}} P^2}{r}} - 1} \right]
\]

This equation relates the 'limiting slenderness ratio' \( \lambda_k \) to the elastic second-order forces \( M \) and \( P \) and to the sectional properties \( l, M, L, r, f_{\text{max}} \) of the actual structure. Sections for which the axial member force is small compared to the bending moment can be assessed by the simplification

\[
\lambda_k = \frac{M L}{r f_{\text{max}}}
\]

Symbols

- \( A \) = area of section
- \( d \) = depth of section
- \( E \) = modulus of elasticity
- \( f_{y} \) = specified yield stress
- \( h \) = storey height or height of structure
- \( h_i \) = height of storey for 'limiting frame'
- \( I \) = second moment of area (moment of inertia)
- \( L \) = length of member
- \( L_k \) = length of beam for 'limiting frame'
- \( M \) = bending moment
- \( M_k \) = bending moment of 'limiting frame'
- \( M_p \) = plastic moment capacity of section
- \( m_O \) = \( \sqrt{P/2E} \) = buckling load parameter
- \( n \) = \( I_{(P/2E)} / (L/h) \) = slenderness ratio of column to beam
- \( P \) = axial force
- \( P_F \) = Euler load of column
- \( P_F \) = failure load
- \( P_{cr} \) = column load associated with 'limiting slenderness ratio'
- \( P_c \) = ultimate axial column load in the absence of moment
- \( P_p \) = plastic collapse load of frame
- \( P_O \) = elastic buckling load of frame
- \( P_{pl} \) = elastic buckling load of 'limiting frame'
- \( P_{pl} \) = plastic collapse load of 'limiting frame'
- \( P_{pl} / P_p \) = ratio of elastic buckling load to plastic collapse load for 'limiting frame'
- \( P_r \) = yield load of member
- \( a \) = uniformly distributed ultimate load intensity
- \( r \) = radius of gyration
- \( y \) = distance from centroid of section to extreme fibre

Greek symbols

- \( \alpha \) = reduction factor defined in Eqn 1
- \( \Delta \) = sway displacement
- \( \lambda \) = slenderness ratio of actual member
- \( \lambda_k \) = slenderness ratio of member in 'limiting frame'
- \( \lambda_i H_0,4 \) = slenderness ratio of member if frame is subjected to non-symmetry

Discussion on papers

Written discussion on the papers in this issue will be accepted until 28 February 1985. This, together with the authors' reply, will be published in the August 1985 issue of The Civil Engineer in South Africa, or later.

Such written discussion, which must be submitted in duplicate, should be in the first person present tense, and should be typed in double spacing. It should be as short as possible and should not normally exceed 600 words in length. It should also conform to the requirements laid down in the 'Notes for the Guidance of Authors and Contributors' as published in the October 1983 issue of The Civil Engineer in South Africa.

Reference

Whenever reference is made to the above this publication should be referred to as The Civil Engineer in South Africa and the volume and date given thus: Civ Engr South Afr, Vol 26, No. 12, 1984.

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594 THE CIVIL ENGINEER in South Africa — December 1984